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INFERENTIAL PROCEDURES IN STABLE DISTRIBUTIONS FOR CLASS FREQUENCY DATA ON INCOMES[1]

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This paper discusses inferential procedures for the family of stable distributions, when the data are tabulated in the form of interval frequencies. The estimation criteria used are minimum chi-square and multinomial maximum likelihood. In evaluating the theoretical probabilities corresponding to the intervals, use is made of the inversion theorem for characteristic functions. Chi-square tail probabilities for independent samples are pooled by means of the Kolmogorov statistic. As an illustration, the methods are applied to Dutch and Australian income data.

1. INTRODUCTION

IN TWO THEORETICAL ARTICLES in the early sixties [**23, 24**] Benoit Mandelbrot argued that an income distribution follows a so-called Pareto-Lévy law, that is, a maximally skew, stable distribution with characteristic exponent α between 1 and 2. So far, this family of distributions has not received much attention from empirical econometricians[2] due to the estimation problems involved. As a rule, the density of a stable distribution cannot be expressed by a simple formula involving elementary functions. Instead, the distributions are described by means of their characteristic functions. This causes rather serious estimation problems which only recently have been solved.[3] Most of the estimation methods proposed assume the availability of individual point data[4] and, hence, cannot be used in case the data have the form of interval frequencies.

The aim of the present paper is to discuss inference methods for the family of stable distributions in the case of interval frequency data. We demonstrate the possibility of estimating parameters of stable distributions by means of (asymptotically) efficient methods (minimum chi-square and multinomial maximum likelihood). We also discuss the possibilities of certain test procedures used for

[1] An earlier and much more extensive version of this paper [**36**] was presented at the Summer Workshop of the Econometric Society at CORE and at the Geneva meeting of the Econometric Society, September, 1978. The authors are indebted to several participants for helpful suggestions. In addition, we wish to thank two referees for several constructive remarks. Remaining errors are ours.

[2] A recent exception is Seastrand [**33**] who uses a method described in Paulson, Holcomb, and Leitch [**26**] based on point data. A survey of the earlier empirical literature on income distributions has been given by Cramer [**3**]. Some recent contributions can be found in Salem and Mount [**32**], Singh and Maddala [**34**], and Kloek and van Dijk [**15, 16**].

[3] Fama and Roll [**6**] and DuMouchel [**4**] confine themselves to the case of symmetric stable distributions. Press [**29**], Leitch and Paulson [**17**], and Paulson, Holcomb, and Leitch [**26**] include the asymmetric case but make use of the sample characteristic function which cannot be computed without gross approximation errors in case of interval frequency data. The only alternative to our method, as far as we know, is given in DuMouchel [**5**]. His estimation criterion is derived from information theory, while we start from classical criteria, viz. minimum chi-square and multinomial maximum likelihood.

[4] The distinction between interval frequency data and individual point data is not exhaustive. There is a third interesting possibility when both interval frequencies and interval means are available. In such a case more efficient estimation is possible; compare Smith [**35**]. Application of that approach to the family of stable distributions is an interesting possibility for further research.

model selection. Some empirical results are reported in which use is made of Australian and Dutch data.[5]

The order of discussion is as follows. In Section 2 we summarize the main properties of the family of stable distributions; in particular, we discuss the interpretation of the parameters. In Section 3 we describe the estimation and test procedures proposed. By way of an illustration we present some of our empirical results in Section 4. Section 5 contains our conclusions. Some technical details are given in the Appendix.

2. STABLE DISTRIBUTIONS

The family of stable distributions was first described by Lévy.[6] The simplest way to describe what is meant by the term *stable* is the following. Let X_1 and X_2 denote independent random variables with a common distribution (apart from location and scale parameters). If their sum X_1+X_2 has the same distribution (apart from location and scale)[7] we say that this distribution is stable.[8] Well-known examples are the normal and Cauchy distributions. In most other cases characteristic functions are the most convenient way to describe stable distributions. Such a characteristic function $\phi(t)$ can be written in the form[9]

$$\ln \phi(t) = \ln E(e^{itX}) = iat - \lambda |t|^{\alpha} \exp\{\tfrac{1}{2} i\pi\gamma \operatorname{sgn}(t)\} \tag{1}$$

for $-\infty < a < \infty$, $\lambda > 0$, $0 < \alpha \leqslant 2$ and $\alpha \neq 1$, $|\gamma| \leqslant 1 - |1-\alpha|$.

The interpretation of the parameters is as follows:

(i) a is a location parameter. If $1 < \alpha \leqslant 2$ the mean exists and equals[10] a.

(ii) Let $\lambda = \delta^{\alpha}$, with $\delta > 0$. Then δ is a scale parameter of the distribution of $(X-a)$. The variance only exists in the particular case $\alpha = 2$; it then equals $2\delta^2$. In

[5] More extensive empirical evidence is given in van Dijk and Kloek [**36**].

[6] See Lévy [**18**]. Discussions of its properties may also be found in Gnedenko and Kolmogorov [**8**], Lukacs [**20**], Loève [**19**], and Feller II [**7**]. A much more concise, but clear discussion is given by Press [**30**].

[7] From now on we shall omit this qualification.

[8] Mandelbrot's argument leading to a stable distribution for incomes is based on the fact that the various income concepts are related to one another by addition: total income of a person may be obtained by adding income from various sources and family income by adding the incomes of the members of the family. Mandelbrot [**23**, pp. 85, 86] assumes that all these types of income (both components and aggregates) follow the same type of distribution (this need not be true for all countries; compare Kestenbaum [**13**]). If one adds an independence assumption, the straightforward consequence is that incomes must follow a stable distribution. The independence assumption is not always realistic, as a referee rightly remarked. However, an obvious generalization for the case of dependent stably distributed random variables runs as follows. Let X_1 and U be independent random variables with a common stable distribution and let $X_2 = \theta X_1 + U$. Then X_1, X_2 and X_1+X_2 have the same stable distribution, while X_1 and X_2 are dependent.

[9] Apart from the case $\alpha = 1$, in which another expression is available, this formula describes all possible stable distributions. The representation adopted here follows Lukacs [**20**, p. 137], with the exception of the sign of γ. Some other authors give different representations. Transformation formulas describing the transition from one representation to another are also given by Lukacs. The mean and the characteristic exponent are the same in all representations. The differences refer to scale and skewness only. It should be noted that the characteristic function (1) also defines distributions in the case $\alpha > 2$, but these do not have the stability property mentioned above.

[10] This property is given little attention in the literature. See [**8**, p. 182] and [**7**, pp. 215 and 528].

that case the distribution is normal. In all other cases the variance is infinite. Nevertheless δ plays the role of a scale parameter in the sense that, if X has a stable distribution, $X^* = (X-a)/\delta$ has a standard stable distribution which depends on α and γ, but is independent of a and δ.

(iii) γ is a skewness parameter. If $1 < \alpha \leqslant 2$, we have $|\gamma| \leqslant 2-\alpha$. If $\gamma = 0$, the distribution is symmetric. If $\gamma = 2-\alpha$, the distribution has maximal positive skewness. According to Mandelbrot [**23**] this restriction defines the relevant subfamily for the description of income distributions. His argument is that otherwise the probability that income is negative will be too large [**23**, pp. 86, 87]. He introduced the term Pareto-Lévy distribution to distinguish this subfamily. Notice that no skewness can occur for $\alpha = 2$ and that the possibilities for skewness are limited for α values close to two.

(iv) α is called the characteristic exponent. It characterizes the tails. For large values of x the density approximates the Pareto density $f(x) = \alpha x_0^\alpha / x^{\alpha+1}$ (if $x > x_0$) with the same parameter α. As in the case of the Pareto distribution the stable distributions have no first- or higher-order integer moments for $0 < \alpha \leqslant 1$; for $1 < \alpha < 2$ the first-order moment exists, but the second does not.[11] Graphical illustrations of the densities of three standard Pareto-Lévy distributions for $\alpha = 1.2$, 1.5, and 1.8 can be found in Mandelbrot [**23**, p. 88]. It is seen that the right-hand tail is heavier as α is smaller. This is in accordance with the facts that the light-tailed Normal family is a subfamily of the stable family with $\gamma = 0$, $\alpha = 2$, and that the heavy-tailed Cauchy family is another subfamily with $\gamma = 0$, $\alpha = 1$.

So far, we considered the distribution of income itself. There is, however, another possibility, suggested by the fact that the normal family is a particular subfamily of the stable family. And the normal family has often been proposed to describe the distribution of *log* income. The underlying theoretical models usually interpret the level of log income as a sum of independent random shocks on which the central limit theorem can be applied (see [**3**, Chapter 4] and the references cited there). Empirical results suggest that the lognormal family is not flexible enough to fit the data (see [**1**, **3**, **35**]). Our own earlier results [**15**, **16**] point in the same direction. So, an obvious generalization is to replace the normal distributions in the random shock models for log income by stable distributions, since the central limit theorem has been generalized for this class of distributions. This leads to the hypothesis that income follows a log stable distribution, or equivalently that log income follows a stable distribution. The latter need not be positively skewed. It may also be symmetric or negatively skewed. The interpretation of the parameters is the same as above if we consider the distribution of log income. If we define a geometric mean (GM) for the continuous case by $GM(y) = \exp(E(\log y))$, $\exp(a)$ may be interpreted as the geometric mean of income.

Finally, we discuss the problem of inequality measures. If one describes an income distribution by a set of parameters, the question may be raised whether one of them can be interpreted as a measure of inequality. For the family of stable

[11] Unlike the case of the Pareto distribution, the second-order moment does exist for $\alpha = 2$.

distributions in general this is a complicated problem as inequality is influenced by δ, α, and γ. The Pareto-Lévy case allows a simplification since $\gamma = 2 - \alpha$, and a further simplification is possible by comparing two Pareto-Lévy distributions with the same[12] α. In that case the ratio δ/a of the scale parameter δ and the mean a is a straightforward measure of relative inequality. In such a comparison it is proportional with several traditional measures such as the relative mean deviation and the Gini coefficient of concentration. For the log stable family the situation is similar, but not quite the same. The simple comparison in this case is between two distributions with the same[13] α and γ. In that case the scale parameter δ has the interpretation of a relative inequality measure.

3. INFERENTIAL PROCEDURES

In this section we describe the estimation and test procedures used. Our first problem is to estimate the parameters of stable distributions when the available data are of the class frequency type. Thus our starting point is a random sample of n individual observations on income which are recorded as frequencies[14] corresponding to m mutually exclusive and exhaustive income intervals. The limits of these intervals are supposed to be exogenously given.[15]

As in our earlier papers [**15**, **16**] we apply two estimation criteria: minimum chi-square (MCS) and multinomial[16] maximum likelihood (MML). Both criterion functions depend on the parameters *via* the theoretical probabilities of the income classes. The estimates are computed by means of numerical optimization and the probabilities by numerical integration. As no closed form for the density is available, this integration is based on the inversion theorem for characteristic functions. Details are given in Appendix A. Both estimation methods are consistent and asymptotically efficient[17] [**31**, pp. 352, 363]. For large samples the two methods are equivalent [**12**, p. 438]. Most of the samples we studied in our applications [**36**] were large enough to yield estimates from both methods which are not far apart.

The MCS method is linked to the chi-square test for goodness of fit. Since the chi-square values are not comparable if the numbers of degrees of freedom are different, we compute the corresponding tail probabilities,[18] which are comparable. The joint hypothesis that a number N of independent data sets have been drawn from distributions belonging to the same family can be tested by adding the

[12] Our empirical evidence [**36**] suggests that for most Pareto-Lévy cases studied the hypothesis $\alpha = 1.5$ can be accepted at least at the 0.01 level.

[13] In many of our empirical examples, the hypothesis $\alpha = 1.8$, $\gamma = 0$ can be accepted.

[14] Such frequencies need not be the result of grouping of individual point data. See footnote 25.

[15] In our application this is true for most intervals, but in the tail classes some endogenous pooling was necessary. We discussed this problem in [**15**, Section 4].

[16] For a discussion of the distinction between ordinary ML and MML, see [**12**, p. 457]. In this context, see also [**2**, **31**].

[17] According to Rao's criterion of second-order efficiency MML is to be preferred [**31**, pp. 348 and 353].

[18] In [**15**, **16**] we used the term "critical level" for the same concept. Harrison [**10**] uses the term "marginal significance level."

chi-square values and at the same time adding the numbers of degrees of freedom. The results obtained in this way appear to be highly sensitive to outliers. So we experimented in [**36**] with the Kolmogorov test [**12**, pp. 468–476]. More precisely, we used $D_n^+ = \sup_x \{S_N(x) - x\}$ where $S_N(x)$ is the empirical distribution function of the chi-square tail probabilities (x). This statistic appears to be somewhat less sensitive to one subgroup having a different distribution than the others. Such a situation can easily occur due to institutional peculiarities or to errors of measurement. For that reason one should avoid rejecting a hypothesis which is acceptable for $N-1$ cases but not for the Nth. The Kolmogorov statistic is based on the tail probabilities mentioned above. Under the null hypothesis, these are asymptotically uniformly distributed on the interval (0, 1). So the test is an approximate one.

The ML method for the parametric version of the multinomial model has been extensively discussed by Rao [**31**, pp. 359–366]. We use minus the inverse of the Hessian of the log likelihood as an approximation for the covariance matrix.[19] This is simpler from a programming point of view than computing the covariance matrix of the MCS estimates.[20]

In our empirical applications we give special attention to the right hand tail classes of the distributions. Consider the null hypothesis that the log stable family is the correct family to be used. Let the disturbances be denoted by $\varepsilon_j = n_j - np_j(\theta)$, where θ is the unknown true parameter vector. Then $\varepsilon_j n^{-\frac{1}{2}}$ is asymptotically normally distributed with zero mean and variance $\sigma_j^2 = p_j(1-p_j)$. Let $\tau_j = \varepsilon_j / \sigma_j n^{\frac{1}{2}}$ denote a standardized disturbance. If we average a number m of independent standardized disturbances we have (approximately) a normal random variable with zero mean and variance $1/m$. This procedure may be applied to the tail disturbances for mutually exclusive samples, which are independent by assumption. In practical applications we replace the disturbances by the observed residuals. This amounts to an additional approximation, so that the results will have to be interpreted with caution if the sample sizes are moderate or small.

If the stable hypothesis is correct for income, the density of the implied distribution for log income has an exponential tail and finite moments. Under the log stable hypothesis, on the contrary, it has infinite variance (if $\alpha < 2$). So, if the stable hypothesis is correct, one might expect that the log stable density will produce too large theoretical probabilities and hence negative residuals in the right hand tail class. An analogous argument may be given for the log stable null hypothesis.

[19] The standard errors in [**36**] show a great deal of variability. (There is one notable exception: the standard errors for the scale parameter δ of the log stable distribution are uniformly very small.) From the theory of linear estimation we know that standard errors tend to be smaller as the fit is better, the sample size is larger, and the degree of multicollinearity is smaller. When we compare the (relative) size of the standard errors with the tail probabilities (a measure of goodness of fit) and the sample sizes, we see that we need the analogue of multicollinearity to explain the results. Compare [**36**, Section 5] and [**16**, Section 5].

[20] The latter is a particular case of minimum distance estimation [**22**, Chapter 9], as is shown in [**11**, pp. 355–356].

So far we have concentrated on null hypotheses and (hence) on type I errors. Of course, it is more satisfactory to consider alternative hypotheses and type II errors as well. The Cox test is a good way to do so. We have given some examples of experiments with the Cox test in [**16**, Section 5]. In these cases the Cox test largely produced the same conclusions as the simple chi-square test. Given the complications inherent to the stable family, application of the Cox test to the present problem would entail a computationally very heavy burden.

Finally, we make a few remarks about measures for goodness of fit in models of the present type.[21] Of course, the chi-square values measure relative goodness of fit: the fit is better as chi-square is smaller, apart from corrections for degrees of freedom. There is an implicit correction in the computation of the tail probabilities: the fit is better as the tail probability is greater. It is true that for a correctly specified model the tail probability is as likely below 0.5 as above it. But if model A is correctly specified, while model B is not, the tail probability of B will likely be less[22] than that of A. Of course one could try to construct some analogue of R^2. But since the model $n_j = np_j(\theta) + \varepsilon_j$ has no constant term one has to face the same problems as in linear models with no constant term.[23]

4. SOME EMPIRICAL RESULTS

We have applied the methods described in the previous section to Dutch and Australian data. Extensive results have been presented in [**36**].[24] For considerations of space we shall confine ourselves to a brief description.

The Dutch data were taken from samples drawn in 1973. The income concept measured is gross income,[25] while families with two or more income earners were left out of consideration. The sample was subdivided into four occupation groups. For the data and for more comments, we refer to [**15**]. The Australian data were published and discussed in [**27**]. They formed the basis for our earlier study [**16**], where different theoretical distributions were considered. The data originate from a survey of consumer expenditure and finances in 1966–1968 and the income concept measured is family disposable income. They have been partitioned in several ways, namely, according to age, occupation, education (of the head of the family), and family size.

A simple way to give summary descriptions for goodness of fit of the Pareto-Lévy and log stable families is by means of the Kolmogorov statistic, which was briefly discussed in Section 3. The results are presented in Table I. The corresponding results for the log t and the Champernowne families, reported in our

[21] Several discussants of the first version and a referee asked for R^2 values.

[22] We shall not try to spell out a formal proof of this statement. The situation is, however, in a sense analogous to that in [**14**] where a similar statement is proved for the linear model.

[23] Compare Maddala [**21**, p. 108].

[24] This paper is available from the authors on request.

[25] It is interesting that the respondents were shown a list containing a set of intervals and that the question asked was: Could you indicate the interval to which your income belongs? It was felt that such a procedure results in a larger number of correct answers. This advantage may offset to some extent the efficiency loss in estimation due to censoring; compare DuMouchel [**5**].

TABLE I

KOLMOGOROV STATISTICS

Data	N	Pareto-Lévy	Log Stable	Log t	Champernowne	Critical Values[c] 5 per cent	1 per cent
1. Dutch Occupations	4	.409	.556	.553	.514	.565	.689
2. Australian Age groups	9	.478[a]	.306	.534[b]	.475[a]	.387	.480
3. Australian Occupations	7	.492[a]	.460[a]	.548[b]	.530[a]	.436	.538
4. Australian Occupations[d]	6	.421	.412	.500[a]	.482[a]	.468	.577
5. Australian Educations	7	.468[a]	.417	.676[b]	.525[a]	.436	.538
6. Australian Family sizes	3	.967[b]	.339	.625	.746[a]	.636	.785

[a] Rejected at a five per cent level of significance, with sample size N, in a one sided test.
[b] Rejected at a one per cent level of significance.
[c] See [**25**]. Note that our statistic is denoted there by D_n.
[d] In this case, one group ("not in work force") is deleted.

earlier papers [**15**, **16**] are presented for comparison. It is seen from Table I that the log stable family is rejected only once in the Kolmogorov tests. This is due to an extremely poor fit of one occupation group which can be identified as the group consisting of people not in the work force. When this group is deleted and the third line of Table I is replaced by the fourth, it is seen that the log stable hypothesis is never rejected, while all other families are rejected several times. It is also seen from Table I that in all cases but one the log stable family yields better fits than the other families. In the pooled right-hand tail residuals test, described in Section 3, all results were insignificant under the log stable null hypothesis, while the Pareto-Lévy hypothesis was rejected for the Australian occupations and family sizes. Closer inspection of the residuals[26] shows that in all seven cases where the chi-square tail probability was less than two per cent, the residuals of the highest income class were negative, while almost all residuals of the second, third, and fourth highest income classes were positive. This looks like rather strong evidence in favor of the log stable family.

The point estimates for α in the Pareto-Lévy case were between 1.35 and 1.68 for most of the data sets, but markedly lower for the Dutch self-employed and old-aged. Probably these groups are more heterogeneous than most other groups considered. For the log stable family most of the point estimates for α were between 1.62 and 1.90, which is larger than in the case of the Pareto-Lévy family (as was to be expected) but smaller than the limiting case $\alpha = 2$, where the distribution is lognormal. Since the lognormal family is a subfamily of the log stable family, a likelihood ratio test could be applied. The lognormal hypothesis was rejected in twenty one cases out of thirty one.

5. CONCLUSIONS

In this paper we demonstrated the possibility of estimating the parameters of stable distributions when the data are tabulated in the form of class frequencies.

[26] See [**36**, Table I and Appendix B].

We emphasized the methodological aspects. For detailed empirical results reference is made to [**36**].

To estimate the parameters of the fitted distributions we made use of the inversion theorem for characteristic functions.

The problem of ascertaining which of the laws fit income better was treated by making use of chi-square tail probabilities and Kolmogorov statistics.

No distribution uniformly gave the best fit but the Pareto-Lévy did reasonably well and the log stable family gave the best fit in the larger number of cases.[27]

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APPENDIX

SOME TECHNICAL DETAILS OF THE ESTIMATION METHOD USED

Let x denote either income or log income and (g_j, h_j) the jth interval. We are interested in the computation of the theoretical probabilities

$$p_j(\theta) = P[g_j \leqslant x < h_j]. \tag{2}$$

These can be obtained from the characteristic function by means of the inversion theorem. Dropping subscripts one obtains

$$p(\theta) = \frac{1}{\pi}\int_0^\infty \frac{1}{t}\,\mathrm{Im}\,\{(e^{-itg} - e^{-ith})\phi(t)\}\,dt; \tag{3}$$

see [**19**, p. 188]. This expression can be rewritten as

$$p(\theta) = \int_0^\infty \frac{1}{\pi t}\exp(-A_1(t))[\sin A_2(h, t) - \sin A_2(g, t)]\,dt \tag{4}$$

where

$$A_1(t) = (\delta t)^\alpha \cos \tfrac{1}{2}\pi\gamma, \tag{5}$$

$$A_2(h, t) = ht - at + (\delta t)^\alpha \sin \tfrac{1}{2}\pi\gamma. \tag{6}$$

The numerical evaluation of (4) was carried out by making use of standard numerical integration[28] on t.

The computed probabilities $p_i(\theta)$ are used in the evaluation of the chi-square and likelihood criterion functions. For optimization we made use of a direct search procedure due to Powell[29] [**28**]. This is an unconstrained optimization method. Since the parameters α and γ are restricted we made use of the transformations

$$\alpha = 1 + |\cos \alpha^*|, \quad \gamma = (2-\alpha)\cos \gamma^*, \tag{7}$$

where α^* and γ^* can take any real value. So the search procedure was done on α^* and γ^*.

[27] More or less similar results for American point data were found by Seastrand [**33**].

[28] We also tried the Bergström-Feller series expansions (see [**20**, Theorem 5.8.2] or [**7**, XVII, 6 Lemma 1]) but these turned out to give considerable approximation errors for low values of α and high values of the standard variable $X^* = (X-a)/\delta$.

[29] Given the availability of a subroutine for Powell's method, this choice has the advantage that it saves programming time, since only the maximand is required, not its derivatives. Other optimization methods may lead to saving CPU time. This trade-off may lead to different decisions under different circumstances.

Standard errors of the estimates were determined by evaluating minus the inverse of the Hessian of the log likelihood function; compare [**31**, p. 366]. The evaluation of the Hessian was done numerically by means of the finite difference method [**9**, pp. 18–21].

The results obtained in this paper required a fair amount of computer time. We think that due to expected improvements in both computer technology and software the cost of computations like these will rapidly decrease.

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